

On Instanton Calculations of $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory

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Abstract

Instanton calculations are demonstrated from a viewpoint of twisted topological field theory. Various properties become manifest such that perturbative corrections are terminated at one-loop, and norm cancellations occur between bosonic and fermionic excitations in any instanton background. We can easily observe that for a suitable choice of Green functions the infinite dimensional path integration reduces to a finite dimensional integration over a supersymmetric instanton moduli space.

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1 Introduction

In $\mathcal{N} = 2$ supersymmetric quantum field theories an simple way is given of how to determine the low energy effective theory quite precisely.[1, 2] Holomorphy and symmetries severely constrain the possibilities of quantum corrections. It was also suggested that the prepotential of low energy effective theory solely arises from contributions of one-loop perturbations and instantons.[3]

An analysis indicated that there exists one instanton contributions.[3] Using factorization property of gauge invariant operators, this contribution is calculated from a two-point Green function $\langle \text{tr}\phi^2(x)\text{tr}\phi^2(y) \rangle$ and the result is shown to be consistent with its direct calculation of $\langle \text{tr}\phi^2 \rangle$ in the weak coupling limit.[4] Here, ϕ is the Higgs field. This indicates that Green functions can be calculated in the weak coupling limit. Upon this observation many calculations of instanton contributions have been done in the weak coupling limit.[5, 7, 6, 8] Those results indicates that non-perturbative contributions seem to be saturated only by instantons.

In the present paper $\mathcal{N} = 2$ supersymmetric Yang-Mills theory is analyzed in terms of twisting and instanton calculations. We easily show that calculations in the weak coupling limit are guaranteed for a class of gauge invariant operators, and non-perturbative contributions are indeed saturated only by instantons. Seen from a viewpoint of twisted topological theory,[9] miraculous cancellations between bosonic and fermionic excitations occur for the class of Green functions. Ultimately and undoubtedly, an infinite dimensional path integration reduces to a finite dimensional integration over a supersymmetric instanton moduli space. With these observations at hand, we demonstrate a one-instanton calculation in terms of the twisted topological field theory. Twisting is a powerful tool to reveal various properties of $\mathcal{N} = 2$ supersymmetric instantons.

We work in a Coulomb branch where a gauge symmetry breaking occurs. In this type of theories instantons do not appear in a simple way. Some zero modes due to instantons are raised by the Higgs mechanism.[10, 11, 12] In general an self-dual instanton solution and the super-partner are specified by a positive integer k and have $8k$ degrees of freedom, respectively. After the gauge symmetry breaking, $8k - 4$ of such zero modes are raised, which we call quasi-zero modes.

This paper is organized as follows. In section 2 we show basic ingredients of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Especially, a viewpoint of topological field theory is introduced. Norm cancellations of massive modes are explained in section 3.

In section 4 we demonstrate a instanton calculation using the twisted topological field theory. Summary and Discussion are found in the last section.

2 $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory

In this section we show basic ingredients of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Especially, a viewpoint of topological field theory is introduced. This is utilized in supersymmetric instanton calculations, and is powerful tool to reveal various properties.

2.1 the Lagrangian

We consider supersymmetric $SU(2)$ Yang-Mills theory whose Lagrangian respects the $\mathcal{N} = 2$ supersymmetry with vanishing central charge. The Lagrangian has a global isospin symmetry $SU(2)_I$. The supercharges Q_α^i and $\bar{Q}_{\dot{\alpha}j}$ transform in the fundamental representation of $SU(2)_I$. The Yang-Mills gauge field A_m is embedded in the $\mathcal{N} = 2$ chiral multiplet \mathcal{A} consisting of one $\mathcal{N} = 1$ chiral multiplet $\Phi = (\phi, \psi_\alpha)$ and one $\mathcal{N} = 1$ vector multiplet $W_\alpha = (\lambda_\alpha, A_m)$. The $\mathcal{N} = 2$ chiral multiplet is arranged as a diamond form

$$\begin{array}{ccc} & A_m^a & \\ \lambda_\alpha^a & & \psi_\alpha^a \\ & \phi^a & \end{array} \quad (2.1)$$

to exhibit the $SU(2)_I$ symmetry which acts on the rows. The $\mathcal{N} = 2$ supersymmetry determines all the relations between the kinetic and interaction terms up to a gauge coupling constant. All the fields are introduced so that the supersymmetry algebra does not contain the gauge coupling constant. We postpone introducing the gauge coupling constant until carrying out path integrations. In $\mathcal{N} = 1$ superspace, the Lagrangian is

$$\mathcal{L} = \int d^4\theta \bar{\Phi} e^{-2V} \Phi e^{2V} + \frac{1}{4} \int d^2\theta W^\alpha W_\alpha + \frac{1}{4} \int d^2\bar{\theta} \bar{W}_\alpha \bar{W}^\alpha. \quad (2.2)$$

We suppress gauge group indices, for example $W^{a\alpha} W_\alpha^a = W^\alpha W_\alpha$, otherwise stated. The covariant derivative is defined on adjoint representations by $D_m \phi^a \equiv \partial_m \phi^a + \epsilon^{abc} A_m^b \phi^c$. Our notations will be found by putting the gauge coupling constant as $g = -1$ in a standard book[13].

Let us recall properties known from looking at the supersymmetric theory (2.2).

In terms of $\mathcal{N} = 1$ superfield formalism, only a subgroup of $SU(2)_I$, which is called $U(1)_J$, is a manifest symmetry. The $U(1)_J$ transformation acts on the $SU(2)_I$ doublet in

a diagonal form; $\Psi \rightarrow \Psi(e^{-i\gamma}\theta)$, $W_\alpha \rightarrow e^{i\gamma}W_\alpha(e^{-i\gamma}\theta)$. The symmetries $SU(2)_I$ and $U(1)_J$ are non-anomalous ones existing in the quantum level.

There is also an Abelian symmetry $U(1)_R$, which ensures that the gaugino λ_α and the Higgsino ψ_α are bare massless, acting as $\Psi \rightarrow e^{2i\gamma}\Psi(e^{-i\gamma}\theta)$, $W_\alpha \rightarrow e^{i\gamma}W_\alpha(e^{-i\gamma}\theta)$. The $U(1)_R$ symmetry is generally broken by an anomaly down to a discrete symmetry \mathbf{Z}_{4N_c} . A \mathbf{Z}_2 subgroup of the discrete R symmetry also contains in the $U(1)_J$ symmetry. Namely,

$$U(1)_R : \begin{array}{cc} A_m & A_m \\ \omega\lambda_\alpha & \omega\psi_\alpha \\ \omega^2\phi & \phi \end{array} = U(1)_J : \begin{array}{cc} A_m & A_m \\ \omega\lambda_\alpha & \omega^{-1}\psi_\alpha \\ \phi & \phi \end{array} \quad (2.3)$$

if $\omega^2 = 1$. In this case the global symmetry is $SU(2)_R \times \mathbf{Z}_8/\mathbf{Z}_2$.

The classical potential of the theory (2.2) is

$$V = \frac{1}{2}[\phi, \bar{\phi}]^2. \quad (2.4)$$

This potential has, so-called, D -flat directions in which we have vanishing vacuum energy $V = 0$. The D -flatness implies that the Higgs fields ϕ and $\bar{\phi}$ belong to a Cartan subalgebra of the gauge symmetry $SU(2)_G$ developing a vacuum expectation value

$$\langle \phi \rangle = \frac{1}{2} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad a \in \mathbf{C}^* \equiv \mathbf{C}/\{0\}. \quad (2.5)$$

This form is given after divided by a gauge transformation continuously connecting to unity. The Weyl group of $SU(2)_G$ acts by $a \rightarrow -a$, so a convenient $SU(2)_G$ gauge invariant operator parameterizing the space of vacua $\mathbf{C}^*/\mathbf{Z}_2$ is $u \equiv \langle \text{tr} \phi^2 \rangle$.

In this situation the $SU(2)_G$ gauge symmetry is broken down to an Abelian subgroup $U(1)$ and the global \mathbf{Z}_8 symmetry is broken to \mathbf{Z}_4 . The spontaneously broken \mathbf{Z}_2 acts on u -plane by $u \rightarrow -u$.

2.2 Wick rotation to Euclidean theory

In order to carry out instanton calculations it is necessary that the theory is Wick-rotated to a Euclidean space. The Wick rotation is defined in momentum space so that integrations over the time component p^0 of a momentum changes by rotating $+90$ degrees around the origin in p^0 complex plane. We can make it by $p^0 \rightarrow ip_4$, or equivalently by $t \rightarrow -it$. The gauge field A_m is also Wick-rotated. The time component is Wick-rotated to $A_0(\mathbf{x}, t) \rightarrow iA_0(\mathbf{x}, -it) \equiv iA_4(\mathbf{x}, t)$ so as to form a consistent covariant derivative

$\partial_0 - iA_0 \rightarrow i(\partial_4 - iA_4)$. In addition, the invariant tensors $\sigma^m_{\alpha\dot{\beta}}$ and $\bar{\sigma}^{m\dot{\alpha}\beta}$ are conveniently redefined. Among other prescriptions, we take $\sigma_{m\alpha\dot{\beta}} = (-1, -i\tau^a)$ $\bar{\sigma}_m^{\dot{\alpha}\beta} = (-1, i\tau^a)$, from which we have $\bar{\lambda}\bar{\sigma}^m D_m \lambda \rightarrow i\bar{\lambda}\bar{\sigma}_m D_m \lambda$. We note that the relation $\bar{\sigma}_m^{\dot{\alpha}\beta} = \sigma_m^{\beta\dot{\alpha}}$ holds, but $\bar{\sigma}_m^{\dot{\alpha}\beta} \neq (\sigma_m^{\alpha\dot{\beta}})^*$. Auxiliary fields are also Wick-rotated by $D \rightarrow iD$, and so on, so as to produce δ -functional. After all, the resultant quantities always have lower indices, if any.

The Wess-Bagger notations after the Wick-rotation will be found in the Appendices A and B of [13] by letting $\eta_{mn} \rightarrow -\delta_{mn}$ and $\epsilon^{mnkl} \rightarrow -i\epsilon_{mnkl}$ with $\epsilon_{1234} = +1$; for example, $\eta_{mn}\sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{n\dot{\beta}\beta} \rightarrow -\sigma_{m\alpha\dot{\alpha}}\bar{\sigma}_m^{\dot{\beta}\beta} = -2\delta_{\alpha}^{\beta}\delta_{\dot{\alpha}}^{\dot{\beta}}$.

Then, the Wick-Rotated Lagrangian is

$$\begin{aligned}\mathcal{L}_E &\equiv -\mathcal{L} \\ &= \frac{1}{4}F_{mn}F_{mn} + D_m\bar{\phi}D_m\phi + \frac{1}{2}[\phi, \bar{\phi}]^2 \\ &\quad - \bar{\lambda}_{\dot{\alpha}i}\bar{\sigma}_m^{\dot{\alpha}\alpha}D_m\lambda_{\alpha}^i - \frac{i}{\sqrt{2}}\bar{\phi}\epsilon_{ij}[\lambda^i, \lambda^j] + \frac{i}{\sqrt{2}}\phi\epsilon^{ij}[\bar{\lambda}_i, \bar{\lambda}_j],\end{aligned}\tag{2.6}$$

where $\lambda_{\alpha}^i = (\lambda_{\alpha}, \psi_{\alpha})$. Now, let us find the $\mathcal{N} = 2$ supersymmetry transformations. The $\mathcal{N} = 2$ transformations consist of a $SU(2)_I$ invariant combination of two $\mathcal{N} = 1$ supersymmetry transformations generated by Q_{α}^1 and Q_{α}^2 . In terms of the $\mathcal{N} = 1$ language, the supercharge Q_{α}^1 acts on λ_{α}^1 and λ_{α}^2 as gaugino and Higgsino, respectively, while the supercharge Q_{α}^2 acts on them as Higgsino and gaugino, respectively. (cf. (2.1)) Then, the supersymmetry transformations are given, splitting them into holomorphic and anti-holomorphic parts $\delta = \delta_{\xi} + \delta_{\bar{\xi}}$, by

$$\begin{aligned}\delta_{\xi}A_m &= \bar{\lambda}_{\dot{\alpha}i}\bar{\sigma}_m^{\dot{\alpha}\alpha}\xi_{\alpha}^i, & \delta_{\bar{\xi}}A_m &= -\bar{\xi}_{\dot{\alpha}i}\bar{\sigma}_m^{\dot{\alpha}\alpha}\lambda_{\alpha}^i, \\ \delta_{\xi}\lambda_{\alpha}^i &= -\sigma_{mn\alpha}^{\beta}\xi_{\beta}^iF_{mn} - \xi_{\alpha}^iD, & \delta_{\bar{\xi}}\lambda_{\alpha}^i &= \sqrt{2}\epsilon^{ij}\sigma_{m\alpha\dot{\beta}}\bar{\xi}_{\dot{\beta}j}D_m\phi, \\ \delta_{\xi}\bar{\lambda}_{\dot{\alpha}i} &= \sqrt{2}\epsilon_{ij}\bar{\sigma}_{m\dot{\alpha}\beta}\xi^{\beta j}D_m\bar{\phi}, & \delta_{\bar{\xi}}\bar{\lambda}_{\dot{\alpha}i} &= \bar{\sigma}_{mn}^{\dot{\beta}\alpha}\bar{\xi}_{\dot{\beta}i}F_{mn} + \bar{\xi}_{\dot{\alpha}i}D, \\ \delta_{\xi}\phi &= -\sqrt{2}\epsilon_{ij}\xi^{\alpha i}\lambda_{\alpha}^j, & \delta_{\bar{\xi}}\phi &= 0, \\ \delta_{\xi}\bar{\phi} &= 0, & \delta_{\bar{\xi}}\bar{\phi} &= -\sqrt{2}\epsilon^{ij}\bar{\xi}_{\dot{\alpha}i}\bar{\lambda}_{\dot{\alpha}j},\end{aligned}\tag{2.7}$$

where $D = -i[\phi, \bar{\phi}]$.

2.3 twisting

After the Wick-rotation the Lorentz group of the space-time changes to a compact rotation group K with appropriate actions, which is locally $SU(2)_L \times SU(2)_R$. The connected component of the global symmetry group is $SU(2)_I$ as seen above. Therefore, the theory

has a global symmetry $H = SU(2)_L \times SU(2)_R \times SU(2)_I$. The supercharges Q_α^i and $\bar{Q}_{\dot{\alpha}j}$ transform under H as $(2, 1, 2)$ and $(1, 2, 2)$, respectively.

Instead of the standard embedding of K in H , we can find two alternative embeddings.[9] As for self-dual instanton calculations the following embedding is the natural one. Let $SU(2)_{R'}$ be a diagonal subgroup of $SU(2)_R \times SU(2)_I$ obtained by sending $SU(2)_I$ index i to dotted index $\dot{\alpha}$. We declare the new rotation group be $K' = SU(2)_L \times SU(2)_{R'}$. The supercharges Q_α^i and $\bar{Q}_{\dot{\alpha}j}$ transform under K' as $(2, 2)$ and $(1, 1) \oplus (1, 3)$, respectively. The K' scalar component of the supercharges is interpreted as a BRST charge Q_B . An interesting point is that, even fermions, $SU(2)_I$ doublet fields have integer spin with respect to K' , breaking the spin-statistics balance. Explicitly, we decompose the gaugino doublet λ_α^i into K' irreducible representations as

$$\begin{aligned}\lambda_{\alpha\dot{\beta}} &= \frac{1}{\sqrt{2}}\sigma_{m\alpha\dot{\beta}}\psi_m, \\ \bar{\lambda}_{\dot{\alpha}\beta} &= \frac{1}{\sqrt{2}}(\bar{\sigma}_{mn\dot{\alpha}\beta}\chi_{mn} + \epsilon_{\dot{\alpha}\beta}\eta).\end{aligned}\tag{2.8}$$

Substituting Eq. (2.8) into the Lagrangian (2.6), we have

$$\begin{aligned}\mathcal{L}_E &= \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \\ \mathcal{L}_1 &= \frac{1}{4}F_{mn}F_{mn} - \chi_{mn}(D_m\psi_n - D_n\psi_m) + \frac{i}{\sqrt{2}}\phi\{\chi_{mn}, \chi_{mn}\}, \\ \mathcal{L}_2 &= -\eta D_m\psi_m + D_m\bar{\phi}D_m\phi - \frac{i}{\sqrt{2}}\bar{\phi}\{\psi_m, \psi_m\}, \\ \mathcal{L}_3 &= \frac{1}{2}[\phi, \bar{\phi}]^2 + \frac{i}{\sqrt{2}}\phi\{\eta, \eta\}.\end{aligned}\tag{2.9}$$

The topological BRST transformation δ_B induced by the $\mathcal{N} = 2$ supertransformations is found by putting $\xi = 0$, $\bar{\xi} = \epsilon_{\dot{\alpha}\beta}\rho/\sqrt{2}$, which reads $\delta = \delta_\xi + \delta_{\bar{\xi}} = -\rho\delta_B$. We have the actions of the BRST transformation δ_B take simple forms to satisfy a nilpotency condition $\delta_B^2 = 0$ on gauge invariant operators. We introduce a multiplier of anti-self-dual field H_{mn} with an appropriate BRST transformation laws. In this case the kinetic term of the Yang-Mills gauge field becomes $\frac{1}{4}F_{mn}F_{mn} = \frac{1}{2}H_{mn}H_{mn} + iH_{mn}F_{mn}^- + \frac{1}{4}F_{mn}\tilde{F}_{mn}$. Here,

$F_{mn}^- \equiv \frac{1}{2}(F_{mn} - \tilde{F}_{mn})$. Now, the BRST transformations are

$$\begin{aligned}
\delta_B A_m &= \psi_m , \\
\delta_B \psi_m &= -\sqrt{2} D_m \phi , \\
\delta_B \chi_{mn} &= i H_{mn} , \\
\delta_B H_{mn} &= \sqrt{2} [\chi_{mn}, \psi] , \\
\delta_B \eta &= i[\phi, \bar{\phi}] , \\
\delta_B \phi &= 0 , \\
\delta_B \bar{\phi} &= -\sqrt{2} \eta .
\end{aligned} \tag{2.10}$$

These BRST transformations are nilpotent up to a gauge transformation $\delta_B^2 = -\sqrt{2}\delta_\phi$; e.g., $\delta_B^2 A_m = -\sqrt{2} D_m \phi$. With these BRST transformations at hand, the $\mathcal{N} = 2$ supersymmetric Yang-Mills Lagrangian becomes BRST exact up to a surface term:[9]

$$\begin{aligned}
\mathcal{L}_1 &= -i\delta_B \chi_{mn} \left(iF_{mn}^- + \frac{1}{2} H_{mn} \right) + \frac{1}{4} F_{mn} \tilde{F}_{mn} , \\
\mathcal{L}_2 &= \frac{1}{\sqrt{2}} \delta_B \bar{\phi} D_m \psi_m , \\
\mathcal{L}_3 &= -\frac{i}{2} \delta_B \eta [\phi, \bar{\phi}] .
\end{aligned} \tag{2.11}$$

These BRST exact forms are supported from a manifestation of the $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 2$ supersymmetric Yang-Mills Lagrangian is written as an integration $\int d^4\theta$ over four chiral superfields θ_α^i . One combination of the superfields θ_α^i gives the topological BRST transformation $\int d\theta = \partial/\partial\theta$. Then, the integration over this combination of the superfields ensures that the Lagrangian takes a BRST exact form.

3 Quantization

In this section we will demonstrate path integrations in the twisted topological theory. For a class of Green functions miraculous cancellations occur between bosonic and fermionic excitations. And ultimately, an infinite dimensional path integration reduces to a finite dimensional integration over a supersymmetric instanton moduli space.

3.1 instanton moduli space

An instanton moduli space \mathcal{M}_k for a given instanton number k is defined by the self-dual equation $F_{mn} = \tilde{F}_{mn}$ modulo small gauge transformation which can be continuously deformed to the identity transformation at infinity. For instanton calculations, however, it

is convenient to include degrees of freedom corresponding to global gauge transformations. So, the moduli space \mathcal{M}_k has dimension $8k$ for a generic $SU(2)$ instanton.

Given an instanton solution A_m , we can find a solution $A_m + \delta A_m$ of its infinitesimal deformation by solving

$$\delta F_{mn}^- = (D_m \delta A_n - D_n \delta A_m)^- = 0 . \quad (3.12)$$

For an anti-symmetric tensor X_{mn} we define $X_{mn}^- \equiv \frac{1}{2}(X_{mn} - \tilde{X}_{mn})$. In addition, we are interested in solutions δA_m orthogonal to the directions of gauge orbit. We impose the gauge fixing condition

$$D_m \delta A_m = 0 , \quad (3.13)$$

where the covariant derivative is defined in terms of the instanton background A_m . The condition (3.13) does not exclude modes of “global” gauge transformations $\delta A_m = D_m \theta$ satisfying $D_m D_m \theta = 0$. The number of solutions of Eqs. (3.12) and (3.13) is the dimension of the moduli space $\dim(\mathcal{M}_k)$.

The equations of motion for the field ψ_m are given by differentiating the Lagrangian (2.9) with respect to χ_{mn} and η . If we restrict field configurations to instanton solutions and omitting higher order terms in g^2 , we have

$$\begin{aligned} (D_m \psi_n - D_n \psi_m)^- &= 0 , \\ D_m \psi_m &= 0 . \end{aligned} \quad (3.14)$$

As will be seen in the subsection 3.4, the fermionic zero modes are enough to be evaluated in the weak gauge coupling limit $g \rightarrow 0$. Comparing Eqs. (3.14) with Eqs. (3.12) and (3.13), we have an interesting relation that fermionic zero modes ψ_m are one-form on the instanton moduli spaces; $\psi_m \sim \delta A_m$. We already encounter this type of relation in Eq. (2.10). Namely, the BRST transformation $\delta_B A_m = \psi_m$ gives a tangent vector of the moduli space.

An index theorem tells us that the number of ψ_m zero modes minus the number of χ_{mn} and η zero modes is equal to the dimension of the moduli space $\dim(\mathcal{M}_k)$. In the present case of a generic $SU(2)$ instanton, there are no χ_{mn} and η zero modes. Then, the number of ψ_m zero modes is equal to the dimension of the moduli space.

The Lagrangian has a $U(1)_R$ symmetry which counts the number of gauginos and Higgsinos; ψ has charge $U = 1$, and χ_{mn} and η have $U = -1$. Therefore, anomaly of $U(1)_R$ charge U gives ψ_m fermionic zero modes.

If we neglect the effect of gauge symmetry breaking, the dimension of the instanton moduli space is $8k$ and there are $8k$ fermionic ψ_m zero modes. In the Coulomb branch, we have a gauge symmetry breaking caused by the Higgs vacuum expectation value (2.5). In this branch the dimension of bosonic true zero modes is 4 and there are 4 fermionic ψ_m zero modes.

3.2 norm cancellations

Dadda-DiVechia[14] used a background Feynman gauge but usually a background Landau gauge[10] is used for instanton, as well as super-instanton, calculations. The calculations are not carried out in a supersymmetric manner, since the gauge fixing procedure violates supersymmetry. It is not a trivial fact that in a different gauge the one-loop determinants of bosons and fermions cancel out completely.

For our convenience, in an instanton background a Landau gauge fixing condition

$$D_m A_m = 0 \quad (3.15)$$

is imposed besides the Wess-Zumino gauge. This condition (3.15) implies (3.13). We should hold Eqs. (3.13) and (3.14) to keep a useful relation that ψ_m is a tangent vector of the instanton moduli space. Then, we add a BRST exact gauge fixing term

$$\mathcal{L}_4 = -iB D_m A_m + \bar{c} D_m (\partial_m c - i[A_m, c]) \quad (3.16)$$

to the original Lagrangian (2.9). Notice that the BRST charge in this case is stemmed from the usual gauge transformation, and is different from the topological one (2.10).

If we use only a single topological BRST transformation as in [15, 16], we have $\mathcal{L}_4 = -\delta_B \bar{c} D_m A_m$ up to a null term. In this case we have $\delta_B^2 = 0$. Such nilpotent BRST transformation consist of a linear combination of (2.10) and the usually defined one (e.g. $\delta_B A_m = D_m c$, $\delta_B \psi_m = i\{c, \psi\}$ and so on). We list first a few:

$$\begin{aligned} \delta_B A_m &= \psi_m + D_m c , \\ \delta_B \psi_m &= -\sqrt{2} D_m \phi + i\{c, \psi\} , \\ \delta_B c &= \sqrt{2} \phi + \frac{i}{2} \{c, c\} , \\ &\dots \end{aligned} \quad (3.17)$$

Even if we use these BRST transformations, the following argument does not change since \mathcal{L}_4 takes the same form.

Now, the quadratic part of the Lagrangian is

$$\begin{aligned}
\mathcal{L}_{\text{quad}} &= \frac{1}{2}H_{mn}^2 + iH_{mn}(D_m A_n - D_n A_m - \epsilon_{mnkl}D_k A_l) \\
&\quad - \chi_{mn}(D_m \psi_n - D_n \psi_m - \epsilon_{mnkl}D_k \psi_l) \\
&\quad + D_m \bar{\phi} D_m \phi + \bar{c} D_m D_m c - \eta D_m \psi_m - iB D_m A_m \\
&\equiv \Phi \Delta_B \Phi - \Psi D_F \Psi ,
\end{aligned} \tag{3.18}$$

where $\Phi = (A_m, H_{mn}, \phi, \bar{\phi}, B)$ and $\Psi = (\psi_m, \chi_{mn}, c, \bar{c}, \eta)$. Notice that the covariant derivative is evaluated in a self-dual instanton background.

The operators D_F and Δ_B have non-zero modes defined by $D_F \Psi = \lambda \Psi$ and $\Delta_B \Phi = \lambda^2 \Phi$, which are specified by a non-zero eigenvalue $\lambda \neq 0$. The path integration over such non-zero modes is given by bosonic and fermionic Gaussian integrations with respect to coefficients with which path integration variables Φ and Ψ are expanded by the eigenstates.

To guarantee the norm cancellation in a definite way, the space-time should have a finite volume and the same boundary condition should be imposed on both the fermion field Ψ and the boson field Φ . The finite volume space-time implies that the eigenstates of the operators Δ_B and D_F are countable. The same boundary condition leads to the same spectrum for the two operators.

Under this situation we may expect that the norm cancellations between the bosons Φ and the fermions Ψ for non-zero modes are guaranteed by the quartet mechanism.[17, 18] Namely, if we calculate the Green function $\langle \mathcal{O} \rangle$ of a physical observable satisfying $\delta_B \mathcal{O} = 0$, path integration over such quartet states does not contribute to the Green function. Therefore, we can integrate out any BRST quartet modes by simply throwing away such modes.

Before proceeding to an explanation of norm cancellations, let us show the three quartet multiplets. We split A_m into two parts. The gauge fixing condition (3.15) defines a transverse part $A_m^{(T)}$. Fluctuations along gauge orbit defines a longitudinal part $A_m^{(L)}$, and can be written in terms of a $SU(2)$ gauge group valued function $\varphi(x)$ by $A_m^{(L)} = D_m \varphi$. These two parts are orthogonal to each other:

$$\int d^4x A_m^{(L)} A_m^{(T)} = \int d^4x D_m \varphi \cdot A_m^{(T)} = - \int d^4x \varphi D_m A_m^{(T)} = 0 .$$

The topological BRST symmetry leads to one quartet multiplet $(A_m^{(T)}, \psi_m^{(T)}; \chi_{mn}, H_{mn})$. The usual gauge symmetry defines the usual quartet multiplet $(A_m^{(L)}, c; \bar{c}, B)$, as well as $(\psi_m^{(L)}, \phi; \bar{\phi}, \eta)$. In the present theory the quartet members are seen schematically in Table 1.

$U =$	-2	-1	0	1	2
			$A_m^{(T)}$		
		χ		$\psi_m^{(T)}$	
			H		
				$\psi_m^{(L)}$	
	$\bar{\phi}$				ϕ
		η			
			$A_m^{(L)}$		
		\bar{c}		c	
			B		

Table 1: The three quartets of the twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. U is ghost number which is identical to the $U(1)_R$ charge.

In the case of calculating the partition function, the Gaussian integrations give

$$\frac{\text{Pf}'(D_F/g^2)}{\sqrt{\text{Det}'(\Delta_B/g^2)}} = 1. \quad (3.19)$$

This can be shown explicitly as follows. First of all, coupling constants appearing in the fermion and boson determinants cancel, since we have the same number of the massive degrees of freedom for the fermions and the bosons. So, we can omit the coupling constant in the partition function by setting $g^2 = 1$. Secondly, the integrations over the usual ghosts c and \bar{c} give $\text{Det}'(D_m D_m)$, while the ϕ and $\bar{\phi}$ integrations give its inverse $1/\text{Det}'(D_m D_m)$. So, these modes are canceled. Thirdly, the integrations over the multipliers B and η simply give δ -functionals with gauge fixing conditions $D_m A_m = 0$ and $D_m \psi_m = 0$, respectively. The modes $A_m^{(L)}$ and $\psi_m^{(L)}$ integrate such δ -functionals canceling the results of themselves. Finally, the remaining Lagrangian can be rewritten in terms of a matrix notation $\frac{1}{2}\Phi^T M_B \Phi - \Psi^T M_F \Psi$ where $\Phi = (A_l^{(T)}, H_{mn})$ and $\Psi = (\psi_l^{(T)}, \chi_{mn})$, and

$$M_B = \left(\begin{array}{c|c} 0 & i \overleftarrow{X} \\ \hline iX & 1 \end{array} \right), \quad M_F = \left(\begin{array}{c|c} 0 & -\overleftarrow{X} \\ \hline X & 0 \end{array} \right), \quad (3.20)$$

with $X \equiv \delta_{ln} D_m - \delta_{lm} D_n + \epsilon_{lmnk} D_k$. These matrices have a transpose symmetry property $M_B^T = M_B$ and $M_F^T = -M_F$. Because of gauge fixing conditions, the fields $A_m^{(T)}$ and $\psi_m^{(T)}$ have the same degrees of freedom as H_{mn} and χ_{mn} , respectively. So, the matrix X has a

determinant. Thus, these forms of the matrices allow us to find the relation

$$\sqrt{\text{Det}' M_B} = \text{Pf}' M_F = \text{Det}' X . \quad (3.21)$$

Therefore, we have Eq. (3.19).

This is a special case of norm cancellations between the bosonic and fermionic excitations by supersymmetry in microscopic one-loop calculations.[14] If we work with no instanton background, the relation (3.19) holds for all the small fluctuations.

3.3 partition function and observable

For calculation of any Green functions the result does not change by twistings, since twistings are merely a change of path integration variables, and we do not modify the metric of the original flat space-time. The twisting approach becomes very efficient when we calculate a Green function of operators that are lowest component of a $\mathcal{N} = 2$ chiral multiplet. In terms of the twisted topological field theory, such operators are BRST singlet. We call this type of Green function chiral Green function. If we calculate a chiral Green function, the right twisting given by (2.8) is the appropriate choice. In this case all good properties of the twisted topological field theory can be inherited.

Let us recall properties of the twisted topological field theory. The Lagrangian of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory consists of two terms. The first one takes a BRST exact form $\{Q_B, V\}$, and the other one is a surface term $\frac{1}{4}F_{mn}\tilde{F}_{mn}$. Notice that since the gauge fixing term (3.16) is also BRST exact, adding this does not change the following property. By omitting the surface term, the partition function

$$Z_{\text{TFT}} = \int DA e^{-\frac{1}{g^2}\{Q_B, V\}} \quad (3.22)$$

is of the twisted topological field theory.

The physical interpretation of states changes significantly by twisting. In terms of the topological BRST operator the subsidiary condition $Q_B|\text{phys}\rangle = 0$ picks up the lowest parts of the supermultiplets in the original supersymmetric theory. Then, the physical states of the twisted topological theory will be much smaller than that of the original supersymmetric theory.

Green functions of the topological field theory does not depend on the bare coupling constant if the inserted operator is BRST singlet $\delta_B \mathcal{O} = 0$. Namely,

$$\frac{d}{d\beta} \langle \mathcal{O} \rangle_{\text{TFT}} = \frac{d}{d\beta} \int DA e^{-\beta\{Q_B, V\}} \mathcal{O}$$

$$\begin{aligned}
&= - \int DA \, e^{-\beta\{Q_B, V\}} \{Q_B, V\mathcal{O}\} \\
&= 0 \, ,
\end{aligned} \tag{3.23}$$

where $\beta = 1/g^2$. Then, we can evaluate the Green functions in the weak coupling limit $g^2 \rightarrow 0$.

If observables are singlet with respect to the BRST charge of the right (or left) twisted theory, only the self-dual (or anti-self-dual) instantons contribute. This is seen from the following Eq. (3.25).

3.4 vanishing theorem

The supersymmetric Yang-Mills theory and its twisted topological theory have the same perturbative corrections around a given background field, since the surface term (in our case, $\frac{1}{4g^2}F_{mn}\tilde{F}_{mn}$) of the Lagrangian does not affect small fluctuations. Suppose that we integrate out massive modes leaving instanton zero modes as a background. The Wilsonian effective Lagrangian is given in terms of a coupling expansion series

$$\frac{1}{g^2}\mathcal{L}_{\text{eff}} = \frac{1}{g^2}\mathcal{L}_{\text{tree}} + \mathcal{L}_{1\text{-loop}} + g^2\mathcal{L}_{2\text{-loop}} + O(g^4) \, . \tag{3.24}$$

If supersymmetry is preserved quite precisely, we might expect that the perturbation series terminates at one-loop. However, the gauge fixing condition (3.13) even violates supersymmetry in our case as well as many cases of (super-) instanton calculations. So it is not trivial that only up to one-loop perturbation corrections survive in such super-instanton calculations.

Fortunately, more than one-loop order terms should vanish in our case, since we can take the weak coupling limit $g^2 \rightarrow 0$ by virtue of the topological BRST symmetry. Then, from (3.24) only the tree and one-loop terms are survived. This reminds us with a result by [19] that in a manifestly supersymmetric calculation the $\mathcal{N} = 2$ Yang-Mills theory has only one-loop divergences.

When we consider a Green function of a BRST singlet observable, it can be seen that the quantum corrections are exhausted by saddle points of instantons and the one-loop fluctuations around them. Mathai-Quillen formula[20, 21] shows that small fluctuations cancel around a saddle point up to a sign. A leading term of the topological Lagrangian reads

$$\mathcal{L}_{\text{TFT}} = \{Q_B, V\} = \frac{1}{2}(F_{mn}^-)^2 + \dots \, . \tag{3.25}$$

Since the first term is positive definite, only the self-dual configurations $F_{mn}^- = 0$ contribute to the Green function in the weak coupling limit. The other terms form BRST quartets to cancel the small fluctuations around the instanton saddle points. This is shown explicitly in subsection 3.2. Other contributions than self-dual instantons are suppressed by $\exp(\frac{-1}{2g^2}(F_{mn}^-)^2 + \dots)$.

Once we know that only the fluctuations around self-dual (or anti-self-dual) instantons contribute to a Green function $\langle \mathcal{O} \rangle_{\text{SYM}}$, we can calculate it using the right (or left) twisted topological field theory. We divide the integration region of the gauge field into subspaces specified by a given instanton number. In each subspace the surface term $\frac{1}{4}F_{mn}\tilde{F}_{mn}$ becomes a constant, and the path integration may be carried out using the twisted topological Lagrangian. Then, neglecting the surface term such a calculation gives a result $\langle \mathcal{O} \rangle_{\text{TFT}}^k$ specified by a given instanton number k . As a result, we have

$$\langle \mathcal{O} \rangle_{\text{SYM}} = \sum_{k=0}^{\infty} \langle \mathcal{O} \rangle_{\text{TFT}}^k e^{-\frac{8\pi^2}{g^2}k}. \quad (3.26)$$

We notice that a holomorphy assumption is not introduced in showing the above fact. It is non-trivial if observables do not receive perturbative corrections beyond one-loop in an instanton background, which is now shown.[9, 21]

4 Supersymmetric Instanton Calculations

We work in a Coulomb branch where gauge symmetry breaking occurs. In this type of theories instantons do not appear in a simple way. Some zero modes due to instantons are raised by the Higgs mechanism.[10, 11, 12] We demonstrate a one-instanton calculation in terms of the twisted topological field theory.

4.1 BPST instanton in singular gauge

Let us consider the $k = 1$ instanton moduli space. The collective coordinates are four translations y_m and one dilatation ρ , and it is convenient to include degrees of freedom which correspond to residual three global gauge rotations θ^a . Now, the $k = 1$ instanton has eight degrees of freedom.

The BPST instanton[22, 23] in singular gauge is

$$A_m^a = 2\rho^2 \frac{\bar{\eta}_{mn}^a x_n}{x^2(x^2 + \rho^2)}. \quad (4.27)$$

where we suppress translations and gauge rotation collective coordinates. The corresponding field strength is

$$F_{mn}^a = -4\rho^2 \frac{\bar{\eta}_{mn}^a}{(x^2 + \rho^2)^2} + 8\rho^2 \frac{(\bar{\eta}_{mk}^a x_n - \bar{\eta}_{nk}^a x_m)x_k}{x^2(x^2 + \rho^2)^2} . \quad (4.28)$$

Let us find small fluctuations of collective coordinates satisfying the gauge fixing condition (3.13). A naive variation $\delta A_m / \delta \gamma$ with γ being a collective coordinate does not satisfy the gauge fixing condition, but it can satisfy the condition by making use of a gauge transformation.[24] The proper variations take a good form

$$\begin{aligned} \delta_T A_m^a &= F_{mn}^a \delta y_n , \\ \delta_D A_m^a &= F_{mn}^a \frac{x_n \delta \rho}{\rho} , \\ \delta_G A_m^a &= F_{mn}^a \left(-\frac{1}{2} \bar{\eta}_{nk}^b x_k \delta \theta^b \right) , \end{aligned} \quad (4.29)$$

where δ_T , δ_D and δ_G mean the variations to the translations ($y_m \rightarrow y_m + \delta y_m$), dilatation ($\rho \rightarrow \rho + \delta \rho$) and global gauge rotations ($\theta^a \rightarrow \theta^a + \delta \theta^a$), respectively.

From these expressions we can easily show that they satisfy the gauge fixing condition using the equation of motion $D_m F_{mn} = 0$ and the self-duality of the curvature.

4.2 fermionic zero modes

To find fermionic zero modes a method is available in $k = 1$ instanton case. The sweeping-out technique[25] is that supersymmetry and superconformal transformations yield all the fermionic zero modes. Beyond the $k = 1$ case a method of tensor products is used.

In our case fermionic zero modes are defined by Eqs. (3.14). Let us recall a nice relation $\delta_B A_m = \psi_m$ interpreted that fermionic zero modes form a complete set of tangent vectors on the instanton moduli space. Then, we easily obtain them by replacing variations of the gauge field to Grassmann variables; $\delta y_m \rightarrow \xi_m$, $\delta \rho \rightarrow \zeta$ and $\delta \theta^a \rightarrow \zeta^a$. Thus, from (4.29) we immediately have

$$\psi_m^a = F_{mn}^a \left(\xi_n + \frac{x_n}{\rho} \zeta - \frac{1}{2} \bar{\eta}_{nk}^b x_k \zeta^b \right) . \quad (4.30)$$

We notice that the $\mathcal{N} = 2$ supersymmetry admits to find all of the fermionic zero modes in generic k instantons. Letting $\xi^{(r)}$ be $8k$ fermionic collective coordinates, the solution $\psi_m = \sum_r Z_{rm} \xi^{(r)}$ can be found using a quantity Z_{rm} defined in [26]. (Z_{rm} satisfies (3.14) and $\partial A_m / \partial \gamma_r = Z_{rm} + D_m \Lambda$ with γ_r being a bosonic collective coordinate.)

4.3 Jacobian and measure of the collective coordinates

After integrating out massive modes, integrations over zero modes remain. In an usual formula zero modes are written in terms of collective coordinates. It is very convenient to change variables from zero modes to such collective coordinates in carrying out integrations, which requires Jacobian from zero modes to collective coordinates.

The metric of the Yang-Mills functional space is defined by

$$||\delta A_m^a||^2 \equiv \frac{1}{2\pi} \int d^4x \delta A_m^a(x) \delta A_m^a(x) . \quad (4.31)$$

This metric gives the A_m path-integration a normalization

$$\int DA e^{-\frac{1}{2} \int d^4x A_m^a(x) A_m^a(x)} = 1 . \quad (4.32)$$

Restricting a variation δA_m to that of collective coordinates $\delta_T A_m$, $\delta_D A_m$ and $\delta_G A_m$ in Eq. (4.31), the metrics of the bosonic collective coordinates are obtained as

$$\begin{aligned} ||\delta_T A_m||^2 &= 4\pi(\delta y_m)^2 , \\ ||\delta_D A_m||^2 &= 8\pi\delta\rho^2 , \\ ||\delta_G A_m||^2 &= 2\pi\rho^2(\delta\theta^a)^2 . \end{aligned} \quad (4.33)$$

The integration measures of the collective coordinates are found using a standard formulation that a metric $ds^2 = g_{ij}dx_i dx_j$ leads to a volume form $\sqrt{\det g_{ij}} d^n x$. Then, the path integral measure constrained on the bosonic collective coordinates are

$$DA = 2^7 \pi^4 d^3\theta d^4y d\rho \rho^3 . \quad (4.34)$$

The θ -integration over the $SU(2)$ group gives $\int_{SU(2)} d^3\theta = 8\pi^2$. If we define a unit Haar measure $d\Omega \equiv d^3\theta/(8\pi^2)$, the overall constant of Eq. (4.34) coincides with that of 't Hooft[10] ($2^{10}\pi^6$).

Next, let us derive the measure of the fermionic zero modes. An easy way to do it is that first we regard the variables ξ_m , ζ and ζ^a as commuting variables tentatively, secondly evaluate the metric of the ψ_m functional space to obtain a Jacobian J from such commuting zero modes to collective coordinates, and then the correct Jacobian of the volume form is $1/J$. The metric is given by

$$\begin{aligned} ||\delta\psi_m^a||^2 &\equiv \int d^4x \delta\psi_m^a(x) \delta\psi_m^a(x) \\ &= 2\pi \left[4\pi(\delta\xi_m)^2 + 8\pi\delta\zeta^2 + 2\pi\rho^2(\delta\zeta^a)^2 \right] . \end{aligned} \quad (4.35)$$

This is an expected result from Eqs. (4.33) by virtue of the wonderful relation $\delta_B A_m^a = \psi_m^a$. Then, the integration measure of the fermionic zero modes is

$$D\psi = \frac{d^4\xi d\zeta d^3\zeta}{(2\pi)^4 \times (2^7\pi^4 \rho^3)} . \quad (4.36)$$

Therefore, the desired total measure is

$$DA \times D\psi = \frac{1}{(2\pi)^4} d^4y d\rho d^3\theta \times d^4\xi d\zeta d^3\zeta . \quad (4.37)$$

It is difficult to find Jacobian factors of bosonic and fermionic zero modes separately, and at best those of $k = 2$ instanton are found[26]. Owing to the $\mathcal{N} = 2$ supersymmetry, at least but this is sufficient, the total Jacobian of the bosonic and fermionic zero modes can be found very easily. The metric of collective coordinate $||\delta\psi_m^a||^2$ is deduced from $||\delta A_m^a||^2$ in the same way as that for the above $k = 1$ case. For a generic $\mathcal{N} = 2$ supersymmetric instanton specified by a positive integer k we immediately have a simple result

$$J_{bose} J_{fermi} = \frac{1}{(2\pi)^{4k}} . \quad (4.38)$$

4.4 Higgs field and gauge symmetry breaking

First, let us consider what is determined from a leading (tree level) behavior of the Higgs field. Next, we consider one-loop corrections to the leading contributions, and that's all. In our prescription the gauge coupling constant is a overall factor of the action. All fields can be regarded as order $O(g)$ quantities. Thus, the relevant terms of the Lagrangian consist of up to third order polynomial of fields.

For the Higgs field the leading equation of motion is $D_m D_m \phi = 0$. This has a solution

$$\phi^a = \frac{x^2}{x^2 + \rho^2} \phi_0^a , \quad (4.39)$$

where ϕ_0^a is a constant. (This form reminds us with $\delta_G A_m^a = D_m(\delta\theta^a x^2/(x^2 + \rho^2))$; the solution of $D_m D_m \phi = 0$ can be obtained from $\delta_G A_m^a$). Similarly $D_m D_m \bar{\phi} = 0$ also determines $\bar{\phi}$ with a constant $\bar{\phi}_0^a$. The vacuum expectation values (see (2.5)) of the Higgs fields ϕ and $\bar{\phi}$ determine their behaviors at infinity, which allow us to find $\phi_0^b = (\bar{\phi}_0^b)^* = (0, 0, a)$. These homogeneous solutions parameterize the D -flat directions of the theory.

An self-dual instanton solution and the super-partner specified by a positive integer k have $8k$ degrees of freedom, respectively. Accompanied by the gauge symmetry breaking (2.5), $8k - 4$ of such zero modes are raised, which we call quasi-zero modes. The four true zero modes are overall translations. An identity $-\delta y_n \partial_n A_m^a = (F_{mn}^a - D_m A_n^a) \delta y_n$

implies that the infinitesimal deformation $F_{mn}^a \delta y_n$ is always the true zero mode satisfying Eqs. (3.12) and (3.13), and similarly for the four fermionic zero modes. The massive modes are dilatation $\delta\rho$, gauge rotations $\delta\theta$ and its superpartners $\delta\zeta$, $\delta\zeta^a$. The Higgs kinetic term produces a mass term of the dilatation mode;

$$\begin{aligned} S_{\text{Higgs}} &= \int d^4x \, D_m \bar{\phi}^a D_m \phi^a \\ &= \int d^4x \, \partial_m (\bar{\phi}^a D_m \phi^a) \\ &= 4\pi^2 \bar{\phi}_0^a \phi_0^a \rho^2 . \end{aligned} \tag{4.40}$$

In addition, Yukawa interaction gives mass term of the fermionic ζ and ζ^a modes;

$$\begin{aligned} S_{\text{Yukawa}} &= -\frac{i}{\sqrt{2}} \int d^4x \, \bar{\phi} \{ \psi_m, \psi_m \} \\ &= -2\sqrt{2}\pi^2 \bar{\phi}_0^a \left(2\rho \zeta^a - \frac{1}{2} \rho^2 \epsilon^{abc} \zeta^b \zeta^c \right) . \end{aligned} \tag{4.41}$$

These show that translation modes y_m and super-translation modes ξ_m remain massless.

Finally, let us consider next order effects in perturbation. The rest of the perturbative corrections does not exist as explained before in subsection 3.4. The relevant terms of the Higgs' equation of motion is

$$D_m D_m \phi + \frac{i}{\sqrt{2}} \{ \psi_m, \psi_m \} = 0 . \tag{4.42}$$

The homogeneous solution of this equation is already given. To obtain the inhomogeneous solution all massive modes may be integrated out by simply setting them zero. Here, the fermionic zero modes are now $\psi_m^a = F_{mn}^a \xi_n$. Then, the inhomogeneous solution is[7]

$$\phi_{\text{inh}}^a = \frac{1}{2\sqrt{2}} \xi_m \psi_m^a . \tag{4.43}$$

We can easily check this solution by plugging it into Eq. (4.42).

4.5 a $k = 1$ instanton calculation

Famously, supersymmetry sometimes provides us with remarkable properties.[27] One of them is a topological property, that Green functions becomes space-time independent if they are of lowest components of chiral superfields. This fact enables us to calculate them in the weak coupling limit. (Needless to say, for such a Green function of gauge invariant operators factorizes.)

Usually, we impose a gauge fixing condition which violates supersymmetry. The standard choice (3.15) is indeed the case. Unfortunately, gauge dependence can be involved in

such a Green function if some of the concerned chiral superfields are not gauge invariant operators. No reliable calculation cannot be done for gauge non-invariant quantities.[28] This fact shows that a operator, say, $\langle \text{tr}\phi^2 \rangle$ cannot be considered as a limit $x \rightarrow y$ of $\langle \text{tr}\phi(x)\phi(y) \rangle$. In conclusion, we have to calculate in a gauge invariant form, whereas, we loose to use the topological property[27].

Now, here comes the point. As for gauge invariant operators calculations in the weak coupling limit is indeed guaranteed by the quite remarkable property (3.23). Thus, let us calculate the observable $\langle \text{tr}\phi^2 \rangle$. The path integral formula of the observable $\langle \text{tr}\phi^2 \rangle$ is

$$\langle \text{tr}\phi^2 \rangle = \int DA e^{-\frac{1}{g^2}\mathcal{L}_E} \text{tr}\phi^2 . \quad (4.44)$$

Here we concentrate on a path integration over the quantum fluctuations continuously deformed to a $k = 1$ super-instanton configuration. All the massive modes are gauge-fixed, apart from instanton quasi-zero modes, by the self-dual condition $F_{mn} = \tilde{F}_{mn}$. Then, according to the vanishing theorem, the path integrations over such modes can be carried out by simply substituting fields with their corresponding zero modes or quasi-zero modes. Therefore, the path integration reduces to a finite dimensional integration over an supersymmetric instanton moduli space.[9]

In actual calculations the determinants of bosons and fermions are needed to be regularized. According to [10], we introduce Pauli-Villars regulators with a mass μ for each field. After renormalizations the ultraviolet divergences are absorbed in the bare coupling $g(\mu)$. Physical quantities do not depend on the cutoff parameter μ , so this parameter should appear in a renormalization invariant form. In $k = 1$ instanton background of the $\mathcal{N} = 2$ supersymmetric pure Yang-Mills theory, such form is precisely $\mu^4 \exp(-8\pi^2/g^2) \equiv \Lambda^4$. Then, the super-determinant in a Pauli-Villars regularization together with the instanton action is

$$\frac{\text{Pf}'(D_F)}{\sqrt{\text{Det}'(\Delta_B)}} e^{-\frac{8\pi^2}{g^2}} = \Lambda^4 . \quad (4.45)$$

Let us compile the results obtained in section 4 to calculate the $k = 1$ super-instanton correction to the quantity $\langle \text{tr}\phi^2 \rangle$. First, the inserted operator $\text{tr}\phi^2$ saturates the true fermionic zero modes ξ_m through the inhomogeneous part of the Higgs field (4.43). Only this inhomogeneous part contribute to the $\int d^4\xi$ integration. With the $\int d^4y$ integration we have

$$\int d^4y \int d^4\xi \text{tr}\phi_{\text{inh}}^2 = \frac{1}{8} \int d^4y F_{mn}^a F_{mn}^a = 4\pi^2 . \quad (4.46)$$

The integrations of fermionic quasi-zero modes lead to

$$\int d\zeta d^3\zeta e^{-\frac{1}{g^2}S_{\text{Yukawa}}} = \frac{(2\pi)^4}{g^4} \bar{\phi}_0^a \bar{\phi}_0^a \rho^3 . \quad (4.47)$$

The factor ρ^3 usually appears in an integration measure[10, 24] of bosonic collective coordinates. This difference is stemmed from the definition of fermionic superconformal modes; $\zeta^a \leftrightarrow \zeta^a/\rho$. The ρ integration gives

$$\int d\rho \rho^3 e^{-\frac{1}{g^2}S_{\text{Higgs}}} = \frac{g^4}{2(2\pi)^4 (\bar{\phi}_0^a \phi_0^a)^2} . \quad (4.48)$$

Finally, an overall factor $\int d^3\theta = 8\pi^2$ is multiplied by.

In conclusion, $k = 1$ super-instanton corrections to the vacuum expectation value $\langle \text{tr}\phi^2 \rangle$ is

$$\frac{\Lambda^4}{a^2} . \quad (4.49)$$

This implies that instanton corrections prevent the symmetric phase $a = 0$. Compared with the exact result[1] using a expansion series[29, 30, 31], we find[4, 5, 6, 7]

$$\Lambda = \frac{1}{2}\Lambda_{\text{SW}} . \quad (4.50)$$

The quantity Λ is a dimensional transmutation scale determined by a perturbative one-loop β function. In general, any β function of coupling constants takes the same form up to two-loop order, as far as mass independent renormalization scheme is used. Then, the definition of Λ does not depend on a wide class of renormalization schemes. As for the quantity Λ_{SW} , this is defined as a scale of $u \equiv \langle \text{tr}\phi^2 \rangle$ in which monopoles and dyons become massless. In other words, Λ_{SW} is a zeros of a low energy effective parameter $\text{Im}(\tau_{\text{eff}})$ along real values of u . So, this quantity is not accessible by any instanton calculations of finite number k . Then, the relation (4.50) should be regarded as an input in instanton calculations. In order to compare the result of instanton calculations with a exact result, at least two results with different instanton number k are required.

In [4, 7, 5] $k = 2$ instanton calculations are demonstrated for the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. Consistency is found between the exact result and instanton calculations.

5 Summary and Discussion

A viewpoint of twisted topological theory[9] is introduced for supersymmetric instanton calculations. This viewpoint enables us to observe that non-perturbative contributions are

indeed saturated only by instantons. Physical observables are defined in terms of a BRST charge stemmed from $\mathcal{N} = 2$ supersymmetry, besides they should be gauge invariant. It is shown that Green functions of physical observables can be calculated in the weak coupling limit. As an example we demonstrate such instanton calculation for a quantity $\langle \text{tr} \phi^2 \rangle$.

The BRST exact form for the Lagrangian is supported from a manifestation of the $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 2$ supersymmetric Yang-Mills Lagrangian is written as an integration $\int d^4\theta$ over four chiral superfields θ_α^i . One combination of the superfields θ_α^i gives the topological BRST transformation $\int d\theta = \partial/\partial\theta$. Then, the integration over this combination of the superfields ensures that the Lagrangian takes a BRST exact form.

We concentrate on a path integration over the quantum fluctuations continuously deformed to a supersymmetric instanton configuration. All the massive modes are found to be gauge-fixed, apart from instanton quasi-zero modes, by the self-dual condition $F_{mn} = \tilde{F}_{mn}$. Then, according to the vanishing theorem explained in subsection 3.4, the path integrations over such modes can be carried out by simply substituting fields with their corresponding zero modes or quasi-zero modes. Therefore, the path integration reduces to a finite dimensional integration over an supersymmetric instanton moduli space.

In actual calculations gauge fixing is needed. Dadda-DiVecchia[14] showed norm cancellations in a background Feynman gauge. While, we impose a background Landau gauge (3.15), and this gauge is usually used for supersymmetric instanton calculations. If a gauge fixing procedure violates supersymmetry, norm cancellations are not guaranteed by supersymmetry completely. Then, it is necessary to check that in a different gauge the one-loop determinants of bosons and fermions cancel out completely. We show this in subsection 3.2.

We show that the $\mathcal{N} = 2$ supersymmetry admits to find all of the fermionic zero modes in generic k instanton background.

It is difficult to find Jacobian factors of bosonic and fermionic zero modes separately, and at best those of $k = 2$ instanton are found[26]. We show that owing to the $\mathcal{N} = 2$ supersymmetry, the total Jacobian of the bosonic and fermionic zero modes is found in Eq. (4.38).

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